Duality of quasilocal gravitational energy and charges with nonorthogonal boundaries

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We study the duality of quasilocal energy and charges with nonorthogonal boundaries in the (2+1)-dimensional low-energy string theory. Quasilocal quantities shown in previous work and also some new variables arising from considering the nonorthogonal boundaries are presented, and the boost relations between these quantities are discussed. Moreover, we show that the dual properties of quasilocal variables, such as quasilocal energy density, momentum densities, surface stress densities, dilaton pressure densities, and Neveu-Schwarz charge density, are still valid in the moving observer's frame.

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I. INTRODUCTION

The study of a gravitational system with finite boundaries has some advantages over that with asymptotic falloff behavior like asymptotic flatness. First, generically, treating a gravitational system with a bounded and finite spatial region should be independent of the asymptotic behavior of the gravitational field. Therefore, this kind of study is very useful for developing a theoretical formulation which is irrelevant of the specific asymptotic properties of the system such as asymptotic flatness. Second, if one constructs a gravitational partition function without any inconsistencies by assuming finite boundaries, then the construction of the gravitational partition function is possible only when a system with finite size is stable. For example, the heat capacity for the Schwarzschild black hole is negative if the temperature at the asymptotic region is fixed, and the partition function for the black hole is not consistent. However, if we consider a fixed temperature at a finite spatial boundary, the heat capacity is positive and the partition function is well defined. Third, from the physical viewpoint, one can define the thermodynamics that is appropriate to observers placed at a finite region from a black hole. In these respects, it is meaningful to define thermodynamic quantities appropriately at a finite boundary.

Some years ago, Brown and York studied quasilocal quantities such as the quasilocal energy, angular momentum, and spatial stress through the Hamilton-Jacobi analysis of a gravitational system [1]. These quantities are closely related to the first law of black hole thermodynamics through the path integral formulation of gravitational systems [2]. This formalism was extended to include the most general case of gauge fields coupled to the dilaton gravity in the context of string theories [3], and the temperature, energy, and heat capacity of AdS black holes were studied by use of this formulation in Ref. [4]. The Hamiltonian and entropy in asymptotically flat spacetimes and anti–de Sitter (AdS) space were studied in Ref. [5], and the relevant issues for the two-dimensional black hole [6] and the quasilocal thermodynamics of Kerr-AdS and Kerr–de Sitter spacetimes [7] have also been intensively investigated.

However, the Brown-York quasilocal formulation is based on the assumption that the spacetime foliation is orthogonal to the timelike boundary that describes the quasilocal quantities seen by static observers in a weak gravitational field, and this seems to be a somewhat strong restriction. When one takes into account finite spatial boundaries in a strong gravitational field, the gravitational force acts on each spatial boundary to a different extent. Therefore, in general, the unit normal defined on the hypersurface at a certain time is not orthogonal to the unit normal defined on the finite spatial boundary, and it is too difficult to calculate the quasilocal quantities seen by observers who are falling into a black hole through a quasilocal formulation with orthogonal boundaries. To generalize the formulation and overcome this difficulty, Booth and Mann reformulated the quasilocal analysis in the presence of nonorthogonal boundaries [8], and related work appears in Ref. [9].

On the other hand, in the context of string theory, duality is considered as a symmetry that relates a certain solution to another one. In (2+1)-dimensional low-energy string theory, this duality is more meaningful in that the dual solution of the Bañados-Teitelboim-Zanelli (BTZ) [10] black hole is known as the (2+1)-dimensional charged black string [11]. The duality of the quasilocal quantities between these dual solutions and the quasilocal thermodynamics of a dilatonic gravitational system with orthogonal boundaries was studied in Ref. [12]. The quasilocal energy density and its dual are invariant under the dual transformation while the quasilocal angular momentum density and its dual are interchanged with the quasilocal Neveu-Schwarz (NS) charge and its dual. In addition, the dual invariance between the surface spatial stress density and the dilaton pressure density appears in the combination of both quantities as $\mathcal{E}=\mathcal{E}^d$, $\mathcal{J}_\phi=-(\mathcal{Q}^d)^\phi$, $(\mathcal{Q})^\phi=-\mathcal{J}^d_\phi$, $\mathcal{S}^{ab}\delta\sigma_{ab}+\Upsilon\delta\Phi=\mathcal{S}^{ab}_d\delta\sigma^d_{ab}+\Upsilon_d\delta\Phi^d$, where $\mathcal{E}$, $\mathcal{J}$, $\mathcal{Q}$, $\mathcal{S}^{ab}$, $\mathcal{Y}$, σ_{ab}, and Φ are the quasilocal surface energy density, the quasilocal momentum density, the quasilocal NS charge density, the quasilocal spatial stress density, the quasilocal pressure density, the surface spatial stress tensor, and the dilaton field, respectively.

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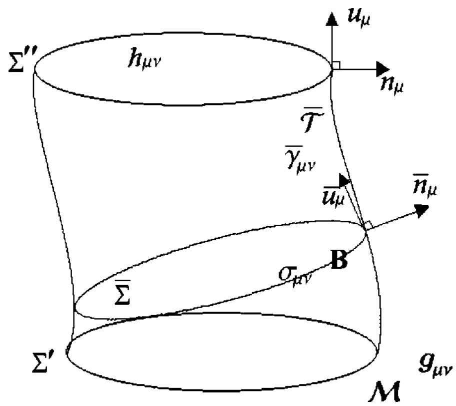

FIG. 1. Spacetime foliation: The spacetime manifold $\mathcal{M}$ that is topologically $\Sigma\times\bar{\mathcal{T}}$ can be foliated by spatial and temporal boundaries denoted by $\bar{\mathcal{T}}$ and Σ, respectively. On each boundary, the unit normal vector, induced metric, and extrinsic curvature are defined.

In this paper, we study the dual properties of quasilocal quantities for $(2+1)$-dimensional dilatonic gravity with nonorthogonal boundaries. In Sec. II, the notation and the setup for the double foliation of quasilocal formalism with nonorthogonal boundaries are presented. The unit vectors normal to both spatial and temporal boundaries are defined, and splittings of the extrinsic curvatures on the spacelike hypersurface and spatial boundary are obtained by the definitions of the induced metrics and the extrinsic curvatures. The quasilocal variables with nonorthogonal boundaries and their boost relations, and dual properties between these variables, are given in Sec. III. In Sec. IV, some concluding remarks and discussion of our results follow.

II. PRELIMINARY: NOTATIONS AND SETUP

In this section, we present a double foliation for Arnowitt-Deser-Misner (ADM) splitting of the metric and some corresponding kinematics. Then we discuss the notation and extrinsic curvature splittings for the quasilocal formalism with nonorthogonal boundaries.

Generically, when we take into account a finite spatial boundary on a manifold $\mathcal{M}$ in a strong gravitational field such as an adjacent region of the black hole horizon, each boundary is exposed to a different gravitational force. This fact enhances the motivation for the generalized quasilocal formalism, which might be possible by considering nonorthogonal boundaries.

Let us consider a double foliation of the spacetime manifold $\mathcal{M}$ with spatial and temporal boundaries as shown in Fig. 1. Then we can take $t=$const and $s=$const surfaces on the boundaries Σ and $\bar{\mathcal{T}}$, and the unit normal vectors are defined as $u_\mu=-N\nabla_\mu t$ on Σ and $\bar{n}_\mu=\bar{M}\nabla_\mu s$ on $\bar{\mathcal{T}}$, where N and $\bar{M}$ are normalization functions determined by satisfying $u\cdot u=-1$ and $\bar{n}\cdot\bar{n}=1$. On the boundaries of Σ and $\bar{\mathcal{T}}$, the induced metrics $h_{\mu\nu}$, $\bar{\gamma}_{\mu\nu}$ and the corresponding extrinsic curvatures $K_{\mu\nu}$, $\bar{\Theta}_{\mu\nu}$ can be defined as

$$h_{\mu\nu}=g_{\mu\nu}+u_\mu u_\nu \quad (\text{on } \Sigma), \tag{1}$$

$$\bar{\gamma}_{\mu\nu}=g_{\mu\nu}-\bar{n}_\mu\bar{n}_\nu \quad (\text{on } \bar{\mathcal{T}}), \tag{2}$$

and

$$K_{\mu\nu}=-h^\alpha_\mu\nabla_\alpha u_\nu \quad (\text{on } \Sigma), \tag{3}$$

$$\bar{\Theta}_{\mu\nu}=-\bar{\gamma}^\alpha_\mu\nabla_\alpha\bar{n}_\nu \quad (\text{on } \bar{\mathcal{T}}). \tag{4}$$

We can define new unit vectors n_μ and $\bar{u}_\mu$ as $n_\mu=MD_\mu s=\gamma^{-1}h^\nu_\mu\bar{n}_\nu$ and $\bar{u}_\mu=-\bar{N}\mathcal{D}_\mu t=\gamma^{-1}\bar{\gamma}^\nu_\mu u_\nu$, where D_μ and $\mathcal{D}_\mu$ are covariant derivatives projected onto the Σ and $\bar{\mathcal{T}}$ surfaces, and the boost factor $\gamma=(1-v^2)^{-1/2}=M/\bar{M}=N/\bar{N}$, where v is a proper radial velocity. From these relations, the relations between unit normal vectors seen in the "barred" and "unbarred" frames are obtained as

$$\bar{u}_\mu=\gamma u_\mu+\gamma v n_\mu,$$
$$\bar{n}_\mu=\gamma n_\mu+\gamma v u_\mu. \tag{5}$$

On the boundary B, the induced metric is given in two ways as $\sigma_{\mu\nu}=g_{\mu\nu}+u_\mu u_\nu-n_\mu n_\nu=g_{\mu\nu}+\bar{u}_\mu\bar{u}_\nu-\bar{n}_\mu\bar{n}_\nu$ and the extrinsic curvatures are also defined as $k_{\mu\nu}=-\sigma^\alpha_\mu\sigma^\beta_\nu\nabla_\alpha n_\beta$ and $\ell_{\mu\nu}=-\sigma^\alpha_\mu\sigma^\beta_\nu\nabla_\alpha u_\beta$. Note that the notation used in this paper for the foliation of spacetimes is summarized in Table I.

On the other hand, the extrinsic curvature on the $\bar{\mathcal{T}}$ boundary, $\bar{\Theta}_{\mu\nu}$, can be split by the extrinsic curvatures on the B boundary, $k_{\mu\nu}$ and $\ell_{\mu\nu}$, to

$$\bar{\Theta}_{\mu\nu}=\gamma k_{\mu\nu}+\gamma v\ell_{\mu\nu}+(\bar{n}\cdot\bar{a})\bar{u}_\mu\bar{u}_\nu + 2\sigma^\alpha_{(\mu}\bar{u}_{\nu)}(n^\lambda K_{\alpha\lambda}-\gamma^2\nabla_\alpha v), \tag{6}$$

where $\bar{a}_\mu=\bar{u}^\alpha\nabla_\alpha\bar{u}^\mu$ is the acceleration of $\bar{u}^\mu$. Similarly, the splitting of the extrinsic curvature on the Σ boundary $K_{\mu\nu}$ is obtained as

TABLE I. Notation for foliation of spacetimes $\mathcal{M}$.

Contents	Metric	Covariant derivative	Unit normal	Intrinsic curvature	Extrinsic curvature	Momentum
Spacetime $\mathcal{M}$	$g_{\mu\nu}$	∇_μ		$R_{\mu\nu\kappa\lambda}$		
Spacelike hypersurface Σ	h_{ij}	D_i	u_μ	$\mathcal{R}_{ijkl}$	K_{ij}	P^{ij}
Timelike hypersurface $\bar{\mathcal{T}}$	γ_{ij}	$\mathcal{D}_i$	$\bar{n}_\mu$		$\bar{\Theta}_{ij}$	$\bar{\Pi}_{ij}$
Boundary $B=\Sigma\cap\bar{\mathcal{T}}$	σ_{ab}				k_{ab}, ℓ_{ab}	

$$K_{\mu\nu}=\ell_{\mu\nu}+(u\cdot b)n_\mu n_\nu+2\sigma^\alpha_{(\mu}n_{\nu)}K_{\alpha\lambda}n^\lambda, \tag{7}$$

by use of the extrinsic curvature on the B boundary, $\ell_{\mu\nu}$, and the acceleration $b^\mu=n^\alpha\nabla_\alpha n^\mu$ of n^μ.

III. DUALITY OF QUASILOCAL QUANTITIES WITH ORTHOGONAL AND NONORTHOGONAL BOUNDARIES

A. Static observers and duality of quasilocal quantities

The dilatonic action coupled with the NS-NS field strength in $(2+1)$ dimensions is given by

$$S=\frac{1}{2\pi}\int_{\mathcal{M}}d^3x\sqrt{-g}\Phi\left[R+\Phi^{-2}(\nabla\Phi)^2+\frac{4}{l^2}-\frac{1}{12}H^2\right]+\frac{1}{\pi}\int_\Sigma d^2x\sqrt{h}\Phi K-\frac{1}{\pi}\int_{\mathcal{T}}d^2x\sqrt{-\gamma}\Phi\Theta, \tag{8}$$

where $-1/2\ln\Phi$ is the dilaton field, H is the three-form field strength of the antisymmetric two-form field B with $H=dB$, and $l^{-2}=-\Lambda$ is a negative cosmological constant.

The variation of the action (8),

$$\delta S=\int_{\mathcal{M}}d^3x\sqrt{-g}[(\Xi_G)_{\mu\nu}\delta g^{\mu\nu}+(\Xi_{\rm dil})\delta\Phi+(\Xi_{\rm NS})^{\mu\nu}\delta B_{\mu\nu}]+\int_\Sigma d^2x[P^{ij}\delta h_{ij}+P_{\rm dil}\delta\Phi+P^{ij}_{\rm NS}\delta B_{ij}]+\int_{\mathcal{T}}d^2x[\Pi^{ij}\delta\gamma_{ij}+\Pi_{\rm dil}\delta\Phi+\Pi^{ij}_{\rm NS}\delta B_{ij}], \tag{9}$$

gives the equations of motion

$$2\pi(\Xi_G)_{\mu\nu}=\Phi G_{\mu\nu}+\nabla_\mu\nabla_\nu\Phi-g_{\mu\nu}\Box\Phi-\frac{1}{2}g_{\mu\nu}\Phi^{-1}(\nabla\Phi)^2-\frac{2}{l^2}g_{\mu\nu}\Phi-\frac{1}{24}g_{\mu\nu}\Phi H^2+\frac{1}{4}\Phi H_{\mu\lambda\sigma}H_\nu^{\lambda\sigma},$$

$$2\pi(\Xi_{\rm dil})=R+\Phi^{-2}(\nabla\Phi)^2-2\Phi^{-1}\Box\Phi+\frac{4}{l^2}-\frac{1}{12}H^2,$$

$$4\pi(\Xi_{\rm NS})^{\mu\nu}=\nabla_\lambda(\Phi H^{\mu\nu\lambda}), \tag{10}$$

where $G_{\mu\nu}=R_{\mu\nu}-1/2g_{\mu\nu}R$ is the Einstein tensor. The conjugate momenta on the Σ and $\mathcal{T}$ boundaries are given by

$$P^{ij}=-\frac{\sqrt{h}}{2\pi}[\Phi(K^{ij}-h^{ij}K)+h^{ij}u^\alpha\nabla_\alpha\Phi],$$

$$P_{\rm dil}=-\frac{\sqrt{h}}{\pi}[\Phi^{-1}u^\alpha\nabla_\alpha\Phi-K],$$

$$P^{ij}_{\rm NS}=\frac{\sqrt{h}}{4\pi}\Phi u^\alpha H^{ij}_\alpha \tag{11}$$

and

$$\Pi^{ij}=\frac{\sqrt{-\gamma}}{2\pi}[\Phi(\Theta^{ij}-\gamma^{ij}\Theta)+\gamma^{ij}n^\alpha\nabla_\alpha\Phi],$$

$$\Pi_{\rm dil}=\frac{\sqrt{-\gamma}}{\pi}(\Phi^{-1}n^\alpha\nabla_\alpha\Phi-\Theta),$$

$$\Pi^{ij}_{\rm NS}=-\frac{\sqrt{-\gamma}}{4\pi}\Phi n^\alpha H^{ij}_\alpha, \tag{12}$$

respectively. In particular, the momenta on the $\mathcal{T}$ boundary are closely related to the quasilocal quantities within this boundary. To specify these quantities, it is useful to decompose the induced metric γ_{ij} into some projections normal to and on the foliation as follows:

$$\delta\gamma_{ij}=-\frac{2}{N}u_iu_j\delta N-\frac{2}{N}u_{(i}\sigma_{j)a}\delta V^a+\sigma^a_{(i}\sigma^b_{j)}\delta\sigma_{ab}, \tag{13}$$

where N is the lapse function and V^a is the shift vector on the $\mathcal{T}$ boundary. As for the NS field B_{ij}, this potential on the boundary $\mathcal{T}$ can be written as $B_{ij}=2u_{[i}\sigma^a_{j]}C_a+\sigma^a_{[i}\sigma^b_{j]}D_{ab}$, where $C_a=\sigma^i_aB_{ij}u^j$ and $D_{ab}=\sigma^i_a\sigma^j_bB_{ij}$ on the B boundary [3], and the variation of B_{ij} produces

$$\delta B_{ij}=\frac{2}{N}u_{[i}\sigma^a_{j]}\delta(NC_a)-\frac{2}{N}u_{[i}\sigma^a_{j]}D_{ab}\delta V^b+\sigma^a_{[i}\sigma^b_{j]}\delta D_{ab}. \tag{14}$$

Putting Eqs. (13) and (14) into the $\mathcal{T}$ boundary term in Eq. (9) leads us to obtain the surface energy density $\mathcal{E}$, the surface momentum density $\mathcal{J}^a$, the spatial stress $\mathcal{S}^{ab}$, the surface NS charge density $\mathcal{Q}^a_{\rm NS}$, the surface NS momentum density $\mathcal{J}^b_{\rm NS}$, and the surface NS current density $\mathcal{I}^{ab}_{\rm NS}$ as

$$\mathcal{E}\equiv-\frac{\delta S_{\mathcal{T}}}{\delta N}=\frac{\sqrt{\sigma}}{\pi}(\Phi k-n^\alpha\nabla_\alpha\Phi), \tag{15}$$

$$\mathcal{J}^a\equiv\frac{\delta S_{\mathcal{T}}}{\delta V^a}=\frac{\sqrt{\sigma}}{\pi}\Phi\sigma^i_aK_{ij}n^j, \tag{16}$$

$$\mathcal{S}^{ab}\equiv\frac{\delta S_{\mathcal{T}}}{N\delta\sigma_{ab}}=\frac{\sqrt{\sigma}}{2\pi}[\Phi(k^{ab}-\sigma^{ab}k+\sigma^{ab}(n\cdot a))+\sigma^{ab}n^\alpha\nabla_\alpha\Phi], \tag{17}$$

$$(\mathcal{Q}_{\rm NS})^a\equiv-\frac{\delta S_{\mathcal{T}}}{\delta(NC_a)}=\frac{\sqrt{\sigma}}{2\pi}\Phi n_i\mathbb{E}^{ia}, \tag{18}$$

$$(\mathcal{J}_{\rm NS})_a\equiv\frac{\delta S_{\mathcal{T}}}{\delta V^a}=(\mathcal{Q}_{\rm NS})^bD_{ba}, \tag{19}$$

$$(\mathcal{I}_{\rm NS})^{ab}\equiv-\frac{\delta S_{\mathcal{T}}}{N\delta D_{ab}}=\frac{\sqrt{\sigma}}{4\pi}\Phi u^\alpha H^{ab}_\alpha, \tag{20}$$

where $\mathbb{E}_{ij}=u^{\lambda}h_i^{\mu}h_j^{\nu}H_{\mu\nu\lambda}$ is the "electric" piece of the three-form field strength.

On the other hand, the equations of motion (10) yields a BTZ black hole solution, which is given by

$$(ds)^2_{\rm BTZ}=-N^2(r)d^2t+f^{-2}(r)d^2r+r^2[d\phi+N^{\phi}(r)dt]^2,$$
$$\Phi=\Phi(r),\quad B_{\phi t}=B_{\phi t}(r), \tag{21}$$

where the lapse function $N^2(r)=(r^2/l^2-M)$, the shift vector $(N^{\phi})^2=J/2r^2$, the dilaton field $\Phi(r)=1$, and the NS two-form field potential $B_{\phi t}(r)=r^2/l$. Duality is a symmetry of string theory, which maps a solution of the low-energy effective string equations with a translational symmetry to another solution [11]. Therefore, this symmetry yields a dual solution of Eq. (21):

$$(ds)^2_d=-N^2(r)d^2t+f^{-2}(r)d^2r+\frac{1}{r^2}[d\phi+B_{\phi t}(r)dt]^2,$$
$$\Phi^d=r^2\Phi,\quad B^d_{\phi t}=N^{\phi}(r), \tag{22}$$

by applying the dual transformation

$$g^d_{xx}=g^{-1}_{xx},\quad g^d_{x\alpha}=B_{x\alpha}/g_{xx},$$
$$g^d_{\alpha\beta}=g_{\alpha\beta}-(g_{x\alpha}g_{x\beta}-B_{x\alpha}B_{x\beta})/g_{xx},$$
$$B^d_{x\alpha}=g_{x\alpha}/g_{xx},\quad B^d_{\alpha\beta}=B_{\alpha\beta}-2g_{x[\alpha}B_{\beta]x}/g_{xx},$$
$$\Phi^d=g_{xx}\Phi, \tag{23}$$

where x is a direction of translational symmetry (ϕ in our case) and the superscript d denotes a dual variable. From Eqs. (21) and (22), the dual properties of the quasilocal energy density, momentum density, and NS charge density are obtained by the straightforward calculation of Eqs. (15), (16), and (18),

$$\mathcal{E}=\mathcal{E}^d=-\frac{f}{\pi}\partial_r(r\Phi),$$
$$\mathcal{J}_{\phi}=-(\mathcal{Q}^d_{\rm NS})^{\phi}=-\frac{r^3f}{2\pi N}\Phi\partial_r N^{\phi},$$
$$(\mathcal{Q}_{\rm NS})^{\phi}=-\mathcal{J}^d_{\phi}=\frac{f\Phi}{2\pi Nr}\partial_r B_{\phi t}. \tag{24}$$

Furthermore, the dual invariance between quasilocal stress density and dilaton pressure density is satisfied by a combination of both quantities as

$$\mathcal{S}^{ab}\delta\sigma_{ab}+\Upsilon\delta\Phi=\mathcal{S}^{ab}_d\delta\sigma^d_{ab}+\Upsilon_d\delta\Phi^d=\frac{f\partial_r N}{2\pi rN}\Phi\delta\sigma_{ab}$$
$$+\frac{f}{\pi}\left(1+\frac{r\partial_r N}{N}\right)\delta\Phi, \tag{25}$$

where the dilaton pressure density is defined as $\Upsilon\equiv N^{-1}\Pi_{\rm dil}$.

B. Moving observers and quasilocal quantities

An extension to the most general case of the quasilocal formalism can easily be established by assuming that the gravitational system has nonorthogonal boundaries as shown in Fig. 1. It amounts to replacing the spatial boundary term $\mathcal{T}$ in the starting action (8) by

$$-\frac{1}{\pi}\int_{\bar{\mathcal{T}}}d^2x\sqrt{-\bar{\gamma}}\Phi\bar{\Theta}. \tag{26}$$

The variation of this action is written like the similar expression of Eq. (9), just by replacing the "barred" expression in the $\bar{\mathcal{T}}$ boundary term, and the boost term $-1/2\pi\int_B dx\sqrt{\sigma}\Phi 2\delta\theta$ is added, where the boost parameter $\tanh\theta=v$. The conjugate momenta on the $\bar{\mathcal{T}}$ boundary are also given as the "barred" variables

$$\bar{\Pi}^{ij}=\frac{\sqrt{-\bar{\gamma}}}{2\pi}[\Phi(\bar{\Theta}^{ij}-\bar{\gamma}^{ij}\bar{\Theta})+\bar{\gamma}^{ij}\bar{n}^{\alpha}\nabla_{\alpha}\Phi],$$
$$\bar{\Pi}_{\rm dil}=\frac{\sqrt{-\bar{\gamma}}}{\pi}(\Phi^{-1}\bar{n}^{\alpha}\nabla_{\alpha}\Phi-\bar{\Theta}),$$
$$\bar{\Pi}^{ij}_{\rm NS}=-\frac{\sqrt{-\bar{\gamma}}}{4\pi}\Phi\bar{n}^{\alpha}H^{ij}_{\alpha}. \tag{27}$$

The ADM splitting of the induced metric on the $\bar{\mathcal{T}}$ boundary is given by the "barred" expression of Eq. (13) while the induced metric h_{ij} on the Σ boundary is split as

$$\delta h_{ij}=\frac{2}{M}n_in_j\delta M+\frac{2}{M}\sigma_{a(i}n_{j)}\delta W^a+\sigma^a_{(i}\sigma^b_{j)}\delta\sigma_{ab}, \tag{28}$$

where M is the lapse function and W^a is the shift vector on the Σ boundary. Putting these splittings of metrics into the boundary actions of Eq. (9) yields

$$\int_{\bar{\mathcal{T}}}d^2x\bar{\Pi}^{ij}\delta\bar{\gamma}_{ij}=-\frac{1}{\pi}\int_{\bar{\mathcal{T}}}d^2x\sqrt{\sigma}\bigg([\Phi(\gamma k+\gamma v\ell)$$
$$-\bar{n}^{\alpha}\nabla_{\alpha}\Phi]\delta\bar{N}-\Phi(\sigma^i_aK_{ij}n^j-\partial_a\theta)\delta\bar{V}^a$$
$$-\frac{\bar{N}}{2}[\Phi(\gamma(k^{ab}-k\sigma^{ab})+\gamma v(\ell^{ab}-\ell\sigma^{ab})$$
$$+(\bar{n}\cdot\bar{a})\sigma^{ab})+\bar{n}^{\alpha}\nabla_{\alpha}\Phi\sigma^{ab}]\delta\sigma_{ab}\bigg) \tag{29}$$

and

$$\int_{\Sigma}d^2xP^{ij}\delta h_{ij}=\frac{1}{\pi}\int_{\Sigma}d^2x\sqrt{\sigma}\bigg((\Phi\ell-u^{\alpha}\nabla_{\alpha}\Phi)\delta M$$
$$-\sigma^i_a\Phi K_{ij}n^j\delta W^a-\frac{M}{2}[\Phi(\ell^{ab}-\ell\sigma^{ab}$$
$$-(u\cdot b)\sigma^{ab})+u^{\alpha}\nabla_{\alpha}\Phi\sigma^{ab}]\delta\sigma_{ab}\bigg). \tag{30}$$

Here we define the quasilocal energy density, the tangential momentum density, and the spatial stress seen by moving observers in the "barred" frame as

$$\bar{\mathcal{E}} = -\frac{\delta S_{\bar{\mathcal{T}}}}{\delta \bar{N}} = \frac{\sqrt{\sigma}}{\pi}[\Phi(\gamma k + \gamma v \ell) - \bar{n}^{\alpha}\nabla_{\alpha}\Phi],$$

$$\bar{\mathcal{J}}_a = \frac{\delta S_{\bar{\mathcal{T}}}}{\delta \bar{V}^a} = \frac{\sqrt{\sigma}}{\pi}\Phi(\sigma^i_a K_{ij} n^j - \partial_a \theta),$$

$$\bar{\mathcal{S}}^{ab} = \frac{\delta S_{\bar{\mathcal{T}}}}{\bar{N}\delta\sigma_{ab}} = \frac{\sqrt{\sigma}}{2\pi}[\Phi\{\gamma(k^{ab} - k\sigma^{ab}) + \gamma v(\ell^{ab} - \ell\sigma^{ab}) + (\bar{n}\cdot\bar{a})\sigma^{ab}\} + \sigma^{ab}\bar{n}^{\alpha}\nabla_{\alpha}\Phi], \tag{31}$$

and the quasilocal normal momentum density, the tangential momentum density, and the temporal stress seen by static observers in the "unbarred" frame as

$$\mathcal{J}_{\vdash} = -\frac{\delta S_{\Sigma}}{\delta M} = -\frac{\sqrt{\sigma}}{\pi}[\Phi\ell - u^{\alpha}\nabla_{\alpha}\Phi],$$

$$\mathcal{J}_a = -\frac{\delta S_{\Sigma}}{\delta W^a} = \frac{\sqrt{\sigma}}{\pi}\Phi\sigma^i_a K_{ij} n^j,$$

$$\Delta^{ab} = \frac{\delta S_{\Sigma}}{M\delta\sigma^{ab}} = -\frac{\sqrt{\sigma}}{2\pi}[\Phi\{\ell^{ab} - \ell\sigma^{ab} - (u\cdot b)\sigma^{ab}\} + \sigma^{ab}u^{\alpha}\nabla_{\alpha}\Phi]. \tag{32}$$

In addition, the dilaton pressure scalar densities on the $\bar{\mathcal{T}}$ and Σ boundaries are calculated as

$$\bar{\Upsilon} = \bar{N}^{-1}\bar{\Pi}_{\text{dil}} = \frac{\sqrt{\sigma}}{\pi}[\Phi^{-1}\bar{n}^{\alpha}\nabla_{\alpha}\Phi - \gamma k - \gamma v\ell + (\bar{n}\cdot\bar{a})],$$

$$\mathcal{Z} = M^{-1}P_{\text{dil}} = -\frac{\sqrt{\sigma}}{\pi}[\Phi^{-1}u^{\alpha}\nabla_{\alpha}\Phi - \ell - (u\cdot b)]. \tag{33}$$

As for the NS charge part, we have the variation of the action for a NS three-form field strength $H_{\mu\nu\rho}$,

$$\delta S_{\text{NS}} = \int_{\mathcal{M}} d^3x\sqrt{-g}(\Xi_{\text{NS}})^{\mu\nu}\delta B_{\mu\nu} + \int_{\Sigma} d^2x P^{ij}_{\text{NS}}\delta B_{ij} + \int_{\bar{\mathcal{T}}} d^2x \bar{\Pi}^{ij}_{\text{NS}}\delta B_{ij}, \tag{34}$$

where the equation of motion $(\Xi_{\text{NS}})^{\mu\nu}$ and the canonical momenta P^{ij}_{NS} and $\bar{\Pi}^{ij}_{\text{NS}}$ on both boundaries are given by Eqs. (10), (11), and (27), respectively. Note that the three-form field strength $H_{\mu\nu\rho}$ is usually decomposed into "electric" and "magnetic" components on a spacelike hypersurface, $\mathbb{E}_{ij} = h^{\mu}_i h^{\nu}_j H_{\mu\nu\rho}u^{\rho}$ and $\mathbb{B} = -\epsilon^{\mu\nu\rho\lambda}H_{\mu\nu\rho}u_{\lambda}/6$, respectively, and it can be shown that $H^2 = 6\mathbb{B}^2 - 3\mathbb{E}_{ij}\mathbb{E}^{ij}$. As shown in Eq. (14), the two-form field potential B_{ij} can be decomposed on the Σ boundary into

$$\delta B_{ij} = -\frac{2}{M}n_{[i}\sigma^a_{j]}\delta(ME_a) - \frac{2}{M}n_{[i}\sigma^a_{j]}D_{ab}\delta W^b + \sigma^{\ a}_{[i}\sigma^{\ b}_{j]}\delta D_{ab}, \tag{35}$$

where $B_{ij} = -2n_{[i}\sigma^a_{j]}E_a + \sigma^a_{[i}\sigma_{j]}{}^b D_{ab}$ and $E_a = \sigma^i_a B_{ij} n^j$, and the field decomposition on the $\bar{\mathcal{T}}$ boundary is written as a similar form of Eq. (14):

$$\delta B_{ij} = \frac{2}{\bar{N}}\bar{u}_{[i}\sigma^a_{j]}\delta(\bar{N}\bar{C}_a) - \frac{2}{\bar{N}}\bar{u}_{[i}\sigma^a_{j]}D_{ab}\delta\bar{V}^b + \sigma^{\ a}_{[i}\sigma^{\ b}_{j]}\delta D_{ab}, \tag{36}$$

where $\bar{C}_{ab} = \sigma^i_a B_{ij}\bar{u}^j$. Hereafter, substituting Eqs. (35) and (36) into Eq. (34) and imposing the equations of motion gives

$$\delta S_{\text{NS}} = \int_{\Sigma} d^2x[-(\mathcal{J}_{\text{NS}})_a\delta W^a - (\mathcal{Q}_{\text{NS}})^a\delta(ME_a) + M(\mathcal{I}^{\text{u}}_{\text{NS}})^{ab}\delta D_{ab}] + \int_{\bar{\mathcal{T}}} d^2x[(\bar{\mathcal{J}}_{\text{NS}})_a\delta\bar{V}^a - (\bar{\mathcal{Q}}_{\text{NS}})^a\delta(\bar{N}\bar{C}_a) - \bar{N}(\bar{\mathcal{I}}^{\text{n}}_{\text{NS}})^{ab}\delta D_{ab}], \tag{37}$$

where the surface NS charge density, the surface NS momentum density, and the surface NS current density seen by static ("unbarred") and moving ("barred") observers are

$$(\mathcal{Q}_{\text{NS}})^a = \frac{\sqrt{\sigma}}{2\pi}\Phi n_i\mathbb{E}^{ia}, \quad (\mathcal{J}_{\text{NS}})_a = (\mathcal{Q}_{\text{NS}})^b D_{ba},$$

$$(\mathcal{I}^{\text{u}}_{\text{NS}})^{ab} = \frac{\sqrt{\sigma}}{4\pi}\Phi u^{\alpha}H^{ab}_{\alpha}, \tag{38}$$

and

$$(\bar{\mathcal{Q}}_{\text{NS}})^a = \frac{\sqrt{\sigma}}{2\pi}\Phi\overline{n_i\mathbb{E}^{ia}}, \quad (\bar{\mathcal{J}}_{\text{NS}})_a = (\bar{\mathcal{Q}}_{\text{NS}})^b D_{ba},$$

$$(\bar{\mathcal{I}}^{\text{n}}_{\text{NS}})^{ab} = \frac{\sqrt{\sigma}}{4\pi}\Phi\bar{n}^{\alpha}H^{ab}_{\alpha}, \tag{39}$$

respectively. Note that the surface NS charge density and the surface NS momentum density in the boosted and unboosted frames are obtained from each boundary term, but the surface NS current densities are divided by two terms projected with respect to the unit normal vectors u^{μ} and $\bar{n}^{\mu}$ in Eqs. (38) and (39). The notation for the quasilocal quantities used in this paper in the boosted and unboosted frames is summarized in Table II.

TABLE II. Notations for quasilocal quantities in boosted and unboosted frames.

Field content	$\frac{\bar{\mathrm{N}}}{\mathrm{N}}$	$\frac{\bar{\mathrm{M}}}{\mathrm{M}}$	$\overline{V^a}$	W^a	σ_{ab}	ME_a	$\overline{NC_a}$	D_{ab}	Φ
Quantities In "barred" frame	$\bar{\mathcal{E}}$	$\bar{\mathcal{J}}_{\vdash}$	$\bar{\mathcal{J}}_a,(\bar{\mathcal{J}}_{\rm NS})_a$		$\bar{\mathcal{S}}^{ab},\bar{\Delta}^{ab}$		$(\bar{\mathcal{Q}}_{\rm NS})^a$	$(\bar{\mathcal{I}}^{\rm n}_{\rm NS})^{ab},(\bar{\mathcal{I}}^{\rm u}_{\rm NS})^{ab}$	$\bar{\Upsilon}$
Quantities In "unbarred" frame	$\mathcal{E}$	$\mathcal{J}_{\vdash}$		$\mathcal{J}_a,(\mathcal{J}_{\rm NS})_a$	$\mathcal{S}^{ab},\Delta^{ab}$	$(\mathcal{Q}_{\rm NS})^a$		$(\mathcal{I}^{\rm n}_{\rm NS})^{ab},(\mathcal{I}^{\rm u}_{\rm NS})^{ab}$	$\mathcal{Z}$

C. Boost relations and duality of quasilocal variables

The quasilocal quantities seen by moving observers are connected to those seen by static observers through the boost relations. We have the quasilocal quantities in the "unbarred" frame as follows:

$$\mathcal{E}=\frac{\sqrt{\sigma}}{\pi}[\Phi k-n^{\alpha}\nabla_{\alpha}\Phi],$$

$$\mathcal{J}_{\vdash}=-\frac{\sqrt{\sigma}}{\pi}[\Phi\ell-u^{\alpha}\nabla_{\alpha}\Phi],$$

$$\mathcal{J}_a=\frac{\sqrt{\sigma}}{\pi}\Phi\sigma^i_a K_{ij}n^j,$$

$$\mathcal{S}^{ab}=\frac{\sqrt{\sigma}}{2\pi}[\Phi(k^{ab}-\sigma^{ab}k+(n\cdot a)\sigma^{ab})+\sigma^{ab}n^{\alpha}\nabla_{\alpha}\Phi],$$

$$\Delta^{ab}=-\frac{\sqrt{\sigma}}{2\pi}[\Phi(\ell^{ab}-\ell\sigma^{ab}-(u\cdot b)\sigma^{ab})+\sigma^{ab}u^{\alpha}\nabla_{\alpha}\Phi], \tag{40}$$

and these are simply converted into the quasilocal quantities seen in the "barred" frame as

$$\frac{\pi}{\sqrt{\sigma}}\bar{\mathcal{E}}=\Phi\bar{k}-\bar{n}^{\alpha}\nabla_{\alpha}\Phi=\Phi(\gamma k+\gamma v\ell)-\bar{n}^{\alpha}\nabla_{\alpha}\Phi,$$

$$\frac{\pi}{\sqrt{\sigma}}\bar{\mathcal{J}}_{\vdash}=-\Phi\bar{\ell}+\bar{u}^{\alpha}\nabla_{\alpha}\Phi$$
$$=-\gamma(\Phi\ell-u^{\alpha}\nabla_{\alpha}\Phi)-\gamma v(\Phi k-n^{\alpha}\nabla_{\alpha}\Phi),$$

$$\frac{\pi}{\sqrt{\sigma}}\bar{\mathcal{J}}_a=\Phi\sigma^i_a\bar{K}_{ij}\overline{n^j}=\Phi(\sigma^i_a K_{ij}n^j-\partial_a\theta), \tag{41}$$

using the relations of the unit normal vectors in Eq. (5). The spatial and temporal stress tensors are given by

$$\frac{2\pi}{\sqrt{\sigma}}\bar{\mathcal{S}}^{ab}=\Phi(\overline{k^{ab}}-\sigma^{ab}\bar{k}+(\overline{n\cdot a})\sigma^{ab})+\sigma^{ab}\bar{n}^{\alpha}\nabla_{\alpha}$$
$$=\gamma[\Phi\{k^{ab}-\sigma^{ab}k+(n\cdot a)\sigma^{ab}\}+\sigma^{ab}n^{\alpha}\nabla_{\alpha}\Phi]$$
$$+\gamma v[\Phi\{\ell^{ab}-\ell\sigma^{ab}-(u\cdot b)\sigma^{ab}\}+\sigma^{ab}u^{\alpha}\nabla_{\alpha}\Phi]$$
$$+\Phi(\bar{u}\cdot\nabla\theta)\sigma^{ab},$$

$$\frac{2\pi}{\sqrt{\sigma}}\bar{\Delta}^{ab}=-\Phi(\overline{\ell^{ab}}-\bar{\ell}\sigma^{ab}-(\overline{u\cdot b})\sigma^{ab})-\sigma^{ab}\bar{u}^{\alpha}\nabla_{\alpha}\Phi$$
$$=\gamma[-\Phi\{\ell^{ab}-\ell\sigma^{ab}-(u\cdot b)\sigma^{ab}\}-\sigma^{ab}u^{\alpha}\nabla_{\alpha}\Phi]$$
$$+\gamma v[-\Phi\{k^{ab}-k\sigma^{ab}+(n\cdot a)\sigma^{ab}\}-\sigma^{ab}n^{\alpha}\nabla_{\alpha}\Phi]$$
$$+\Phi(\bar{n}\cdot\nabla\theta)\sigma^{ab} \tag{42}$$

by using Eq. (5), $(\overline{n\cdot a})=\gamma(n\cdot a)-\gamma v(u\cdot b)+\bar{u}\cdot\nabla\theta$, and $(\overline{u\cdot b})=\gamma(u\cdot b)-\gamma v(n\cdot a)-\bar{n}\cdot\nabla\theta$. Therefore, the boost relations between the surface energy density, the tangential momentum density, the normal momentum density, the spatial stress, and the temporal stress in the boosted and unboosted frames are obtained as

$$\bar{\mathcal{E}}=\gamma\mathcal{E}-\gamma v\mathcal{J}_{\vdash},$$

$$\bar{\mathcal{J}}_{\vdash}=\gamma\mathcal{J}_{\vdash}-\gamma v\mathcal{E},$$

$$\bar{\mathcal{J}}_a=\mathcal{J}_a-\frac{1}{\pi}\Phi\partial_a\theta,$$

$$\bar{\mathcal{S}}^{ab}=\gamma\mathcal{S}^{ab}-\gamma v\Delta^{ab}+\frac{\sqrt{\sigma}}{2\pi}\Phi(\bar{u}\cdot\nabla\theta)\sigma^{ab},$$

$$\bar{\Delta}^{ab}=\gamma\Delta^{ab}-\gamma v\mathcal{S}^{ab}+\frac{\sqrt{\sigma}}{2\pi}\Phi(\bar{n}\cdot\nabla\theta)\sigma^{ab}, \tag{43}$$

by using Eqs. (40), (41), and (42), and the boost relations for the quasilocal NS charge densities, NS momentum densities, and NS current densities are simply given by

$$(\overline{\mathcal{Q}}_{\rm NS})^a=\gamma^2(\mathcal{Q}_{\rm NS})^a+2\gamma^2v^2n_\mu(\mathcal{I}^{\rm u}_{\rm NS})^{\mu a},$$

$$(\overline{\mathcal{J}}_{\rm NS})_b=\gamma^2(\mathcal{J}_{\rm NS})_b+2\gamma^2v^2n_\mu(\mathcal{I}^{\rm u}_{\rm NS})^{\mu a}D_{ab},$$

$$(\overline{\mathcal{I}}^{\rm n}_{\rm NS})^{ab}=\gamma(\mathcal{I}^{\rm n}_{\rm NS})^{ab}+\gamma v(\mathcal{I}^{\rm u}_{\rm NS})^{ab}, \tag{44}$$

by means of Eqs. (38) and (39). Note that the boost invariance of the tangential momentum density in Eq. (43) and NS charge density in Eq. (44) is straightforwardly calculated for the metric (21) as $(\overline{\mathcal{Q}}_{\rm NS})^\phi=(\mathcal{Q}_{\rm NS})^\phi$ and $\overline{\mathcal{J}}_\phi=\mathcal{J}_\phi$, respectively, which are expected results since the only motion in our case is perpendicular to the angular direction.

Let us now show the duality relations between the surface energy densities $\overline{\mathcal{E}}$ and $\overline{\mathcal{E}}^d$, the tangential momentum densities $\overline{\mathcal{J}}_\vdash$ and $\overline{\mathcal{J}}^d_\vdash$, the normal momentum densities $\overline{\mathcal{J}}_\phi$ and $\overline{\mathcal{J}}^d_\phi$, and the NS charge densities $(\overline{\mathcal{Q}}_{\rm NS})^\phi$ and $(\overline{\mathcal{Q}}^d_{\rm NS})^\phi$. Using the boost relations in Eqs. (43) and (44), the dual relations are given by

$$\overline{\mathcal{E}}=\overline{\mathcal{E}}^d,$$

$$\overline{\mathcal{J}}_\vdash=\overline{\mathcal{J}}^d_\vdash,$$

$$\overline{\mathcal{J}}_\phi=-(\overline{\mathcal{Q}}^d_{\rm NS})^\phi,$$

$$(\overline{\mathcal{Q}}_{\rm NS})^\phi=-\overline{\mathcal{J}}^d_\phi, \tag{45}$$

note that these relations are exactly the same forms as those in the orthogonal boundary case. Notice that Eq. (45) shows that the dual properties between the quasilocal variables are still valid regardless of observers who measure the quasilocal variables in their own frames.

Next let us focus on the dual invariance of the spatial and temporal stress densities and dilaton pressure densities. Basically, the quantity $(n\cdot a)$ has a dual invariance for the metrics (21) and (22), and it yields $(\overline{n}\cdot\overline{a})=\gamma(n\cdot a)=(\overline{n}\cdot\overline{a})_d$. In the ''barred'' frame, the combination of spatial stress and dilaton pressure density satisfies the dual invariance, which is given by

$$\begin{aligned}\overline{\mathcal{S}}^{ab}\delta\sigma_{ab}+\overline{\Upsilon}\delta\Phi&=\gamma(\mathcal{S}^{ab}\delta\sigma_{ab}+\Upsilon\delta\Phi)\\&\quad+\frac{\gamma v}{2\pi}\partial_r\theta\left(\frac{f}{r}\Phi\,\delta\sigma_{ab}+2rf\,\delta\Phi\right)\\&=\gamma(\mathcal{S}^{ab}_d\delta\sigma^d_{ab}+\Upsilon_d\delta\Phi^d)\\&\quad+\frac{\gamma v}{2\pi}\partial_r\theta\left(r^3f\Phi\,\delta\sigma^d_{ab}+\frac{2f}{r}\delta\Phi^d\right)\\&=\overline{\mathcal{S}}^{ab}_d\delta\sigma^d_{ab}+\overline{\Upsilon}_d\delta\Phi^d,\end{aligned} \tag{46}$$

and the additional dual relation for the temporal stress density Δ^{ab} and the dilaton scalar density $\mathcal{Z}$ is obtained by a simple calculation,

$$\Delta^{ab}\delta\sigma_{ab}+\mathcal{Z}\delta\Phi=\Delta^{ab}_d\delta\sigma^d_{ab}+\mathcal{Z}_d\delta\Phi^d=0. \tag{47}$$

As a result, all the quasilocal quantities are reformulated by the double foliation of the quasilocal analysis with nonorthogonal boundaries, and the relevant boost relations are presented. Furthermore, the dual properties for the quasilocal variables are still valid even in the moving observer's frame.

IV. DISCUSSIONS

We have studied the duality of quasilocal energy and charges for the $(2+1)$-dimensional dilatonic gravitational system with nonorthogonal boundaries by use of the double foliation of the spacetime manifold $\mathcal{M}$. The quasilocal variables including the surface energy density, momentum densities, spatial and temporal stresses, and the quantities related to the NS three-form field strength were presented and the dual relations between these quantities were proposed. In this approach, the boosting is confined to the radial direction, so the angular momentum densities and NS charge density are independent of the boost factor γ while the energy density is mixed with the tangential momentum density $\mathcal{J}_\vdash$. In other words, these quantities are naturally expected to have general covariance under Lorentz-type transformations.

On the other hand, for a noncompact spacetime, quasilocal quantities are not well defined in the limit that a finite boundary R goes to infinity. This unexpected inconsistency can be removed by introducing reference background spacetimes with an action S_0, and the physical action can be defined as $S_{\rm phys}=S-S_0$. However, this reference background spacetime action does not guarantee preservation of the covariance of quasilocal quantities, since these variables in the reference background spacetimes will transform with a different velocity compared with the velocity of the quasilocal surface in the given spacetime. Nevertheless, the reference background spacetime action does not alter the dual properties of quasilocal quantities. In fact, the action (8) is reduced to the effective action $S_{\rm eff}=(1/2\pi)\int_{\mathcal{M}}d^3x\sqrt{-g}(R+2/l^2)$ by imposing the solution of the dilaton field and the NS three-form field strength [13]. It evidently describes AdS_3 spacetimes, and the gravitational counterterm for AdS_3 spacetimes can be considered as a reference background spacetime action. For an AdS spacetime, the counterterm action can be constructed by an algorithmic procedure and it is uniquely determined [14]. The counterterm action of Eq. (8) is written as $S_{\rm ct}=-(1/\pi l)\int_{\overline{\mathcal{T}}}d^2x\sqrt{-\overline{\gamma}}\Phi$, where $\Phi(r)=1$ for the BTZ black hole, which is compatible with the action shown in Refs. [14,15], and it is invariant under the dual transformation (23). More precisely, the reference background action gives the reference energy density, the reference spatial stress, and the reference dilaton pressure density as $\overline{\mathcal{E}}_0=\sqrt{\sigma}\Phi/\pi l$, $\overline{\mathcal{S}}^{ab}_0=-\sqrt{\sigma}\sigma^{ab}\Phi/2\pi l$, and $\overline{\Upsilon}_0=-\sqrt{\sigma}/\pi l$, respectively. A short glance at $\mathcal{E}_0$ shows that it is invariant under boosting and dual transformation, i.e., $\overline{\mathcal{E}}_0=\mathcal{E}_0$ and $\overline{\mathcal{E}}_0$

$=\overline{\mathcal{E}}_0^a$. In addition, the combination of $\overline{\mathcal{S}}_0^{ab}\delta\sigma_{ab}+\overline{\Upsilon}_0\delta\Phi$ is also invariant under the dual transformation (23). Therefore, the physical quasilocal quantities obtained by subtracting the values of the reference background spacetimes inevitably satisfy the usual properties of dual transformations for any observers, whether they are moving or not.

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